

Current relaxation in nonlinear random media

Tsampikos Kottos¹ and Matthias Weiss²

¹ *Max-Planck-Institut für Strömungsforschung, Bunsenstrasse 10, D-37073 Göttingen, Germany and*

² *MEMPHYS-Center for Biomembrane Physics, Physics Dept., University of Southern Denmark, Campusvej 55, DK-5230 Odense M, Denmark*

We study the current relaxation of a wave packet in a nonlinear random sample coupled to the continuum and show that the survival probability decays as $P(t) \sim 1/t^\alpha$. For intermediate times $t < t^*$, the exponent α satisfies a scaling law $\alpha = f(\Lambda = \chi/l_\infty)$ where χ is the nonlinearity strength and l_∞ is the localization length of the corresponding random system with $\chi = 0$. For $t \gg t^*$ and $\chi > \chi_{\text{cr}}$ we find a universal decay with $\alpha = 2/3$ which is a signature of the *nonlinearity-induced delocalization*. Experimental evidence should be observable in coupled nonlinear optical waveguides.

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A fundamental source of physical information are time-resolved decay measurements in quantum mechanical systems which are coupled to a continuum via metallic leads or absorbing boundaries. While the radioactive decay is a prominent paradigm, more recent examples include atoms in optically generated lattices and billiards [1, 2], the ionization of molecular Rydberg states [3], photoluminescence spectroscopy of excitation relaxation in semiconductor quantum dots and wires [4], and pulse propagation studies with electromagnetic waves [5]. Motivated by the experimental achievements, the dynamics of open quantum systems has also gained considerable interest from a theoretical perspective and various analytical techniques have been developed to study the problem in more detail [6, 7, 8, 9, 10, 11, 12, 13]. One possible approach to the problem is to consider the survival probability $P(t)$ of a wave packet which is initially localized inside an open sample of size N . The total current leaking out of the sample is then related to the survival probability by $J(t) = -\frac{\partial P(t)}{\partial t}$.

For ballistic/chaotic systems [7], and for random systems in the metallic regime [9], $P(t)$ is by now well understood. Recently, also quantum systems with a mixed classical phase space have been studied [10, 13], where it was found that $P(t) \sim 1/t$. The same algebraic decay was found [10] for disordered (or dynamical) systems with exponential localization [14, 15, 16]. In both cases, this law is related to localization and tunneling effects, and applies for intermediate asymptotic times [10]. For larger times, localization effects lead to a log-normal decay of the survival probability [9].

The subject of the present paper is the survival probability in a new setting, namely a class of *random* systems where the evolution is governed by a *Discrete Nonlinear Schrödinger Equation* (DNLSE) (see [17] and references therein), that is

$$i \frac{\partial}{\partial t} \psi_n(t) = \sum_m H_{nm} \psi_m(t) - \chi |\psi_n(t)|^2 \psi_n(t). \quad (1)$$

Here, $|\psi_n(t)|^2$ denotes the probability for a particle to

be at site n at time t , H_{nm} is a random tight-binding Hamiltonian and χ is the strength of the nonlinearity. The nonlinear term in Eq. (1) can arise due to a mean field approximation for many-body interactions, e.g. in the Gross-Pitaevskii framework for Bose-Einstein Condensates [18], or it can result from the description of a quantum mechanical particle which moves in a random potential and interacts strongly with vibrations [19]. The DNLSE can be also viewed as a description for the energy transfer in proteins [20]. Finally, in the context of optics the DNLSE is capable of describing wave motion in coupled nonlinear optical waveguides [21, 22, 23]. In the latter case the longitudinal space dimension of the waveguide plays the role of the time variable.

Here, we present for the first time the consequences on the decay of the survival probability $P(t)$, i.e. on the flux out of the sample when coupling a nonlinear random system like Eq. (1) to the continuum via absorbing boundaries (or conducting leads). In particular, we find that for intermediate (but large) times

$$P(t) \approx \frac{C}{t^\alpha} \quad \text{with} \quad \alpha(\chi, l_\infty) = f(\Lambda = \frac{\chi}{l_\infty}) \quad (2)$$

where C is some constant and $f(\Lambda)$ is a universal scaling function which encodes the interplay between the nonlinearity and the disorder. We find that $f(\Lambda < \Lambda^*) \approx 1$ while $f(\Lambda \gg \Lambda^*) \approx 0.35 \pm 0.05$ where $\Lambda^* \sim \mathcal{O}(1)$.

Moreover, for nonlinearity strengths χ above the *delocalization border* χ_{cr} at which localization phenomena are destroyed [23], and for times $t > t^* \sim N^{2/\nu}/(\chi l_\infty)^2$ (where $\nu \approx 2/5$ is determined by the asymptotic spreading of the wavepacket's variance $\text{var}(t) \sim t^\nu$), we find a universal asymptotic power-law decay

$$P(t) \approx C \left(\frac{\chi}{t} \right)^{2/3}. \quad (3)$$

The intermediate power law (2) is observed for a large class of localized initial excitations (i.e. δ -like, narrow Gaussians) near the surface of the sample which

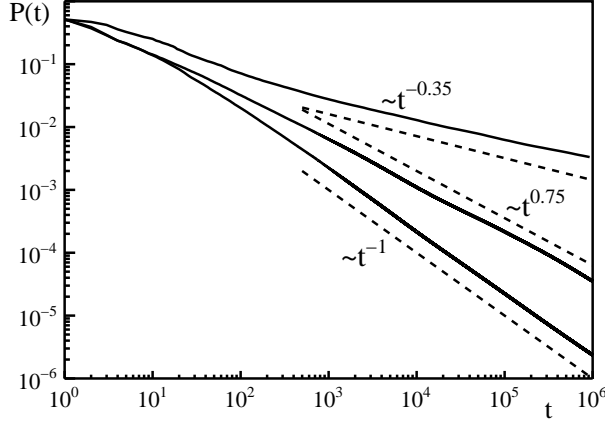


FIG. 1: The survival probability $P(t)$ [Eq. (5)] shows a power-law decay $P(t) \sim t^{-\alpha}$ for intermediate (but large) times $t < t^*$. Shown are three representative examples for $k = 5$, $K = 7$, $N = 1024$ and $\chi = 1, 15, 200$ (bottom to top).

have attracted both experimental and theoretical interest [5, 10, 11, 12, 13] in recent years. On the other hand (3) is independent of the initial condition and applies as soon as the wavepacket is spread ergodically over the sample. Our results (2,3) are confirmed numerically and are supported by theoretical arguments.

To investigate the decay of the survival probability we use a modified version [23] of the Kicked Rotor (KR) with absorbing boundary conditions [10]. Based on the similarities [15, 16] between dynamical and Anderson localization, it is expected [23, 24] that the same dynamics will be generated by Eq. (1). The discrete quantum mechanical evolution from time t to time $t + 1$ (measured in units of a kick period T) is described by the map

$$\psi_n(t+1) = \sum_m e^{i\pi(m-n)} J_{n-m}(k) e^{-iT((m+\phi)^2 - \chi|\psi_m|^2)} \psi_m(t) \quad (4)$$

where the Bessel function J_{n-m} appears as the result of the kick described by the operator $U_{\text{kick}} = \exp(-ik \cos(\theta))$. Here, θ denotes the angle, n is the angular momentum while $K = kT$ is the classical kicking strength. The parameter ϕ can be interpreted as an Aharonov-Bohm flux through the ring parameterized by the coordinate θ . In contrast to the standard KR now the kinetic term depends on the wavefunction probability. The modified KR (4) models propagation of nonlinear waves in optical fibers with a change of the optical density inside the waveguide [22, 23]. The same model approximately describes propagation in waveguides with longitudinal sinusoidal modulation of the boundary [22, 23].

The dynamics of the model (4) for $\chi = 0$ is by now well studied. The classical motion is chaotic and for sufficiently large values of K there is diffusion in momentum space with diffusion coefficient $D \simeq \frac{k^2}{2}$ [15]. The most

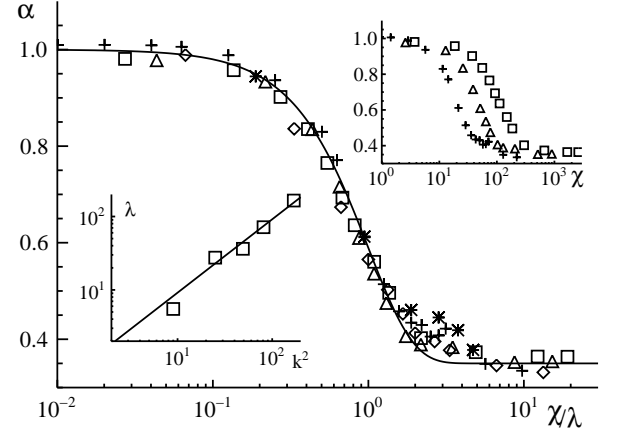


FIG. 2: Power-law exponents α of the survival probability as a function of the scaled nonlinearity strength χ/λ for various localization lengths $l_\infty \approx k^2$ with a heuristic scaling function (full line). Error bars are smaller than the symbol size. Lower inset: The scaling parameter λ increases linearly with l_∞ (full line, slope ≈ 1). Upper inset: Power-law exponents α vs. the nonlinearity strength χ for $k = 5(+), 9(\triangle), 13(\square)$.

striking consequence of quantization is the suppression of this classical diffusion due to quantum-dynamical localization [15], the dynamical version of the well-known Anderson localization [14]. The eigenstates of the associated unitary operator \mathcal{U} are exponentially localized with a localization length $l_\infty \simeq k^2$. The dynamical localization, however, was found to be destroyed for nonlinearity strengths $\chi > \chi_{\text{cr}} \sim 1/T$ [23]. A subdiffusive growth $\text{var}(t) \approx (T\chi l_\infty)^{4/5} t^\nu$ of the wave packet's variance emerged, leading to a uniform spreading over the whole sample. The value $\nu = 2/5$ was deduced in [23] on the basis of the Chirikov criterion of overlapping resonances and was found to be in reasonable agreement with numerical results. For times $t > t^* \sim \frac{N^5}{(T\chi l_\infty)^2}$ the subdiffusion leads to a uniform distribution of an initially localized wavepacket over the sample of size N .

To open the system described by Eq. (4) we additionally apply a projection operator \mathcal{P} to the wavepacket in a fixed interval of momentum states $0 \leq n \leq N$, i.e. $|\tilde{\psi}(t)\rangle = \mathcal{P}|\psi(t)\rangle$. Thus, \mathcal{P} describes the complete deletion (absorption) of the part of the wavepacket which propagates outside the given interval. The survival probability inside the sample at a time t is then given by $p(t) \equiv |\tilde{\psi}(t)|^2 = \langle \tilde{\psi}(t) | \tilde{\psi}(t) \rangle$. To suppress fluctuations, we concentrate on the geometric average of $p(t)$, i.e.

$$P(t) = \exp(\langle \ln(p(t)) \rangle_\phi), \quad (5)$$

where $\langle \cdot \rangle_\phi$ indicates an averaging over different phases ϕ (typically more than 20). The initial excitation (unless stated otherwise) is a δ -like wave packet launched at one of the boundaries, i.e. $\psi_n(t=0) = \delta_{n,0}$. In our

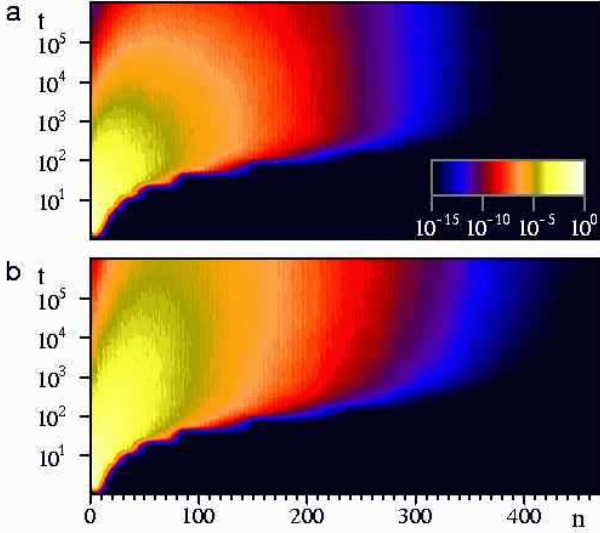


FIG. 3: Density plots of a wave-packet evolution for (a) $\chi = 1$ and (b) $\chi = 50$ (parameters as in Fig. 1). Notice the development of probability depletion near the absorbing boundary for $\chi = 50$ which leads to the creation of a potential barrier $V_\chi = -\chi|\psi_n(t)|^2$. The time axis and the color-coded height of the wave function (see inset in a) are both on a log-scale.

numerical calculations, we have used $K = kT = 7$ and $k = 3, 5, 7, 9, 13$ while the sample length N was chosen to fulfill always $N \gg l_\infty$. Due to localization, the survival probability shows a decay $P(t) \sim 1/t$ for $\chi = 0$ [10].

In Fig. 1 we report the survival probability $P(t)$ for $k = 5$, $N = 1024$ and three representative values of the nonlinear coupling ($\chi = 1, 15, 200$). In all cases, $P(t)$ clearly displays a power-law decay $P(t) \sim t^{-\alpha}$ for times $t < t^*$. The exponents α for various nonlinearities χ and localization lengths l_∞ are summarized in Fig. 2. All curves have the same functional form, albeit being shifted with respect to each other (Fig. 2, upper inset). The curves $\alpha(\chi)$ do coincide, however, when plotting them versus $\Lambda = \chi/\lambda$ where λ is a scaling parameter (Fig. 2, main part). We find that the scaling parameter λ depends linearly on l_∞ (Fig. 2, lower inset) resulting in a one-parameter scaling of the power-law exponent according to Eq. (2). Our data indicate that for $\Lambda < \Lambda^* \approx 0.5$ localization effects are dominant and the survival probability decays as $P(t) \sim 1/t$. In the opposite limit of large nonlinearities the decay of $P(t)$ follows a universal law $P(t) \sim 1/t^\alpha$ with $\alpha \approx 0.35 \pm 0.05$. We found that the smooth transition between the two limits is characterized by the scaling function $f(\Lambda) = 0.35 + 0.65 \exp(-\Lambda^{3/2})$.

In Fig. 3 we report the spatio-temporal evolution of the wave packet for two limiting values of χ in a density plot. For $\chi = 1$ the initial excitation is essentially localized in a region close to the absorbing boundary, covering the interval $0 \leq n \leq l_\infty$ in a uniform manner (Fig. 2a).

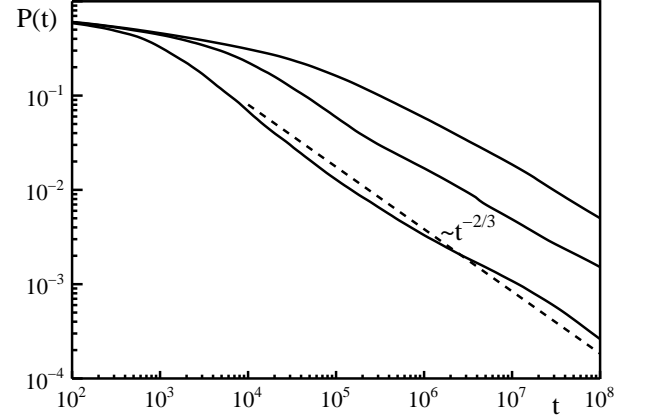


FIG. 4: Decay of $P(t)$ for $N = 256$, $k = 5$, $K = 7$ and $\chi = 2, 10, 50$ (from top); the numerical result is in agreement with Eq. (3) (dashed line). We used a uniform (ergodic-like) excitation as initial condition.

This is the same behavior as found for $\chi = 0$ (data not shown). As the nonlinearity increases, subdiffusion takes over, leading to a spreading of the dominant fraction of the wavefunction into the bulk of the sample. This results in a gradual depletion of the wavefunction from the boundary zone and a creation of an effective potential barrier, the height of which increases with the nonlinearity strength χ as $V_\chi(t) = -\chi|\psi_n(t)|^2$. This barrier traps the probability inside the bulk of the sample and obstructs the outwards flux thus leading to the observed slow decay of $P(t)$ for intermediate times $t < t^*$ [25].

We want to understand the one-parameter scaling of the power-law exponent $\alpha(\chi, l_\infty) = f(\Lambda)$ (see Fig. 2). We recall that the slow decay to the continuum in the presence of the nonlinearity is due to tunneling through a barrier of height $V_\chi \sim \chi|\psi|^2$ which is created in a regime $n \sim l_\infty$ from the boundary. Assuming that for very small nonlinearities the components ψ_n of the wave packet are similar to the case $\chi = 0$, i.e. $|\psi_n| \sim \exp(-n/l_\infty)/\sqrt{l_\infty}$, we obtain $V_\chi \sim \chi/l_\infty$. The barrier is totally transparent if $V_\chi \leq E$ where $E \sim \mathcal{O}(1)$ is the energy of the excitation. Consequently we get that $\Lambda^* \equiv \chi^*/l_\infty \sim \mathcal{O}(1)$ indicating that the ratio $\Lambda = \chi/l_\infty$ controls the behavior of $P(t)$ and therefore the value of α (see Eq. (2)).

Next, we give a qualitative argument for the power-law decay (2) when $\Lambda \lesssim \Lambda^*$. We start with the perturbative expression for the survival probability for $\chi = 0$ [10], i.e.

$$P(t) \sim \int_0^\infty |c_k|^2 e^{-\Gamma_k t} dk, \quad (6)$$

where $|c_k|^2 = |\langle \psi(t=0) | u_k \rangle|^2 \sim \exp(-2k/l_\infty)/l_\infty$ are the overlaps of the initial state $|\psi(t=0)\rangle$ with the exponentially localized eigenstates $|u_k\rangle$ of the operator \mathcal{U} [Eq. (4)] for $\chi = 0$. The decay $P(t) \sim 1/t$ results

from Eq. (6) when using $\Gamma_k \sim T_k \sim \exp(-2k/l_\infty)/l_\infty$, where T_k is the transmission probability of the k -th mode. In the presence of a nonlinearity, it is known that the transmission probability still decays exponentially [24, 26], and therefore it is reasonable to assume that $T_k \sim \exp(-2k/l_\chi)$ where $l_\chi = \alpha l_\infty$ is an effective length determined by both, the tunneling through the barrier V_χ and the disordered potential. Since the existence of the barrier additionally hampers the transmission T_k with respect to $\chi = 0$, it is reasonable to assume that $\alpha < 1$. Substituting this expression for T_k into Eq. (6) we obtain Eq. (2).

For times $t \geq t^*$ and $\chi > \chi_{\text{cr}}$, the subdiffusion leads to an ergodic-like distribution of the initial excitation over the whole sample [23]. A simple approach based on a modified diffusion equation suggests that $P(t)$ should consequently decay in a stretched-exponential manner [27]. It is thus surprising that our data contradict this very expectation (see Fig. 4). In fact, we find a *universal power-law* decay with exponent $\alpha \approx 2/3$. This can be understood by working in the basis of localized eigenstates $|u_k\rangle$ of the linear case ($\chi = 0$). In this representation, we can estimate the transition rate from a quasi-energy level E_k to other levels in a distance $\sqrt{\text{var}(t)}$ to be $\Gamma_k \sim (T\chi)^2/[\text{var}(t)]^{3/2}$ [23]. Thinking semi-classically, the latter can be associated with the escape time $t_k \sim \Gamma_k^{-1}$ of a particle that was initially located at a distance $\sqrt{\text{var}(t)}$ from the absorbing boundary. At any time t the number μ of particles with inverse escape times $\Gamma_k > t^{-1}$ is thus given by $\mu \sim \sqrt{\text{var}(t)} \sim [(T\chi)^2 t]^{1/3}$. As the probability to survive inside the sample up to a time t is given by $P(t) \propto d\mu/dt$ [10], this leads to $P(t) \sim (T\chi/t)^{2/3}$ in agreement with our data (see Fig. 4).

We have checked also that for $\chi < \chi_{\text{cr}}$ localization effects are dominant leading to a decay of $P(t)$ as in the case of $\chi = 0$ [9, 10]. Finally, in the limit $t \rightarrow \infty$, $|\psi_n(t)|^2$ becomes very small due to the loss of norm from the absorbing boundaries and $P(t)$ decay as for $\chi = 0$ [9, 10].

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[25] We confirmed numerically that the exponent ν of $\text{var}(t) \sim t^\nu$ is independent of χ on time scales on which we observe Eq. (2); this also was true for the exponent of the return probability $P_s(t) = |\langle \psi_0 | \psi(t) \rangle|^2$. Therefore, subdiffusion alone cannot explain the dependence $\alpha(\chi)$.
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take localization corrections into account [5, 8, 9].